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The extension of Boadway's transformation technique to two or more dimensional moving boundary problems

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1. INTRODUCTION

NUMEROUS technical problems involve the movement of a phase boundary induced by the diffusion of energy or mass. Most common examples including the conduction of heat are the solidification of casting, the thawing of permafrost, the freezing of foods, aerodynamic heating of missiles and in many other geophysical problems. Mathematically, these problems belong to so-called moving boundary problems in which the moving interface divides the relevant field into at least two regions. Such problems become nonlinear because the location of the moving interface is not known a priori. Due to this nonlinearity analytical solutions can be found only in limited situations, for example, as in Neumann's solution for a one-dimensional problem. Also, the vast majority of theoretical work in this area has been limited to the analysis of one-dimensional moving boundary problems.

To date, several methods are available for the solution of two-dimensional moving boundary problems. In most cases, the emphasis has been placed on a general class of two-dimensional solidification or melting problems, and the following discussion is thus given in the context of this kind of system. Surveys of the early literature with numerous references dating from the time of Stefan are given in Crank's [1] comprehensive book; in Fasano and Primicerio [2] is contained an up-to-date account of mathematical developments and of wide ranging applications to problems in physical and biological sciences, engineering, metallurgy, soil mechanics, decision and control theory, etc. The present note proposes a relatively simple numerical method for the solution of multi-dimensional moving boundary problems on extending and modifying Boadway's [3] transformation. The idea in the present scheme is a particular case of the curvilinear transformation, that is one in which the dependent variable is interchanged with one of the space variables. The variation of this method, the so-called Isotherm Migration Method (IMM) was proposed by Chernousko [4] and independently by Dix and Cizek [5] and subsequently developed and extended to two space dimensions by Crank and co-workers [6–9] and Turland [10]. The use of coordinate transformation for immobilizing the boundary in the case of two-dimensional moving boundary problems has been reported by some of the authors. For example, Furzeland [11] used body-fitted curvilinear coordinate transformation for transforming a curve-shaped region into a fixed rectangular domain; Saitoh [12] and Duda *et al.* [13] discussed several problems using polar coordinates together with the immobilization transformation. More recently, Sparrow and Hsu [14] used coordinate transformations for a control volume formulation. Finally, in their formulation, Gupta and Kumar [15] gave a method based on coordinate transformation which transformed the time varying domain into an invariant one.

In our approach, we propose a relatively simple numerical method for the multi-dimensional moving boundary problems by an independent variable interchange. The present scheme is an extension of Boadway's [3] transformation to time-dependent moving boundary problems in a two or more dimensional case. In our method, not only the shape of the moving interface but also that of the fixed boundary can be selected arbitrarily, thereby allowing its application to more practical situations regardless of the geometry of the problem considered.

2. THE EXTENSION OF BOADWAY'S TRANSFORMATION

For the purpose of illustration, governing equations are presented for the two-dimensional case, since the extension to three dimensions is accomplished in a similar manner. Hence, a particular case of the curvilinear transformation for the heat flow equation can be performed for example, following Boadway's [3] treatment of fluid flow problems. The equation for heat flow in a homogeneous medium in which the heat conductivity k and specific heat c may be functions of temperature U, and density ρ, can be written as

$$c\rho\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left(k\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial U}{\partial y}\right). \quad (1)$$

Here temperature U is a function of space coordinates (x, y) at a given time, i.e.

$$U = U(x, y) \tag{2}$$

hence

$$dU = (\partial U/\partial x)\,dx + (\partial u/\partial y)\,dy. \tag{3}$$

Now we introduce a new dummy variable

$$\phi(x, y) \tag{4}$$

and a similar relation for $d\phi$ as given in equation (3). What is required are new functions

$$x = x(\phi, U) \quad \text{and} \quad y = y(\phi, U) \tag{5}$$

for which we have

$$dx = \left(\frac{\partial x}{\partial \phi}\right) d\phi + \left(\frac{\partial x}{\partial U}\right) dU \tag{6}$$

and similarly for dy. On using equation (6) together with equation (3) and the corresponding expression for $d\phi$, we obtain

$$dx = \frac{\partial x}{\partial \phi}\left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy\right) + \frac{\partial x}{\partial U}\left(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy\right) \tag{7}$$

and similarly for dy. Then by collecting the dx terms in the two expressions we obtain

$$\begin{bmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial U} \\ \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial U} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial U}{\partial x} \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{8}$$

The solution is

$$\frac{\partial \phi}{\partial x} = \frac{\partial y/\partial U}{A} \quad \text{and} \quad \frac{\partial U}{\partial x} = \frac{-\partial y/\partial \phi}{A} \tag{9}$$

where A is the determinant of matrix (8). Similarly collecting dy terms we obtain

$$\begin{bmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial U} \\ \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial U} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial y} \\ \frac{\partial U}{\partial y} \end{bmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{10}$$

from which

$$\frac{\partial \phi}{\partial y} = \frac{-\partial x/\partial U}{A} \quad \text{and} \quad \frac{\partial U}{\partial y} = \frac{\partial x/\partial \phi}{A}. \tag{11}$$

In dealing with second derivatives we obtain

$$\frac{\partial}{\partial x}\left(K\frac{\partial U}{\partial x}\right) = \frac{\partial}{\partial \phi}\left(K\frac{\partial U}{\partial x}\right)\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial U}\left(K\frac{\partial U}{\partial x}\right)\frac{\partial U}{\partial x} \tag{12}$$

in which first derivatives are given by equations (9) and (11). A similar expression can be obtained for $\partial(K\partial U/\partial y)/\partial y$. Finally, we wish to interchange temperature U with one of the space variables say, y, i.e. to write $y = y(x, U)$ and reference to the second term of equation (5) shows that we must now let $\phi = x$ in the above analysis. When this is done in the development of equation (12) and the corresponding expression for $\partial(K\partial U/\partial y)/\partial y$, noting that determinants A of matrices (8) and (10) are reduced to $\partial y/\partial U$, and also that

$$\left.\frac{\partial y}{\partial t}\right|_U = -\left(\frac{\partial y}{\partial U}\right)\left(\frac{\partial U}{\partial t}\right) \tag{13}$$

the heat equation (1) can be written as

$$c\rho\frac{\partial y}{\partial t} = \left(\frac{\partial y}{\partial U}\right)\frac{\partial}{\partial x}\left(k\frac{\partial y}{\partial x}\right) - \frac{\partial k}{\partial U} - \left(\frac{\partial y}{\partial x}\right)\frac{\partial}{\partial U}\left(k\frac{\partial y}{\partial x}\right) \tag{14}$$

where for convenience k denotes $K(\partial y/\partial U)$. If K, c are constants equation (14) reduces to

$$\frac{c\rho\partial y}{k\partial t} = -\left\{\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial U^2}\left(1 + \left(\frac{\partial y}{\partial x}\right)^2\right)\left(\frac{\partial y}{\partial U}\right)^{-2} - 2\frac{\partial^2 y}{\partial x\partial U}\frac{\partial y}{\partial x}\left(\frac{\partial y}{\partial U}\right)^{-1}\right\}. \tag{15}$$

Equation (15) represents y as a function of U, x and t. Turland [10] derived equation (15) on obtaining the complete transformation by using a slightly different argument relevant to the present problem. Finally, similar ideas can be extended to derive the three space dimensional heat conduction equation by introducing the dummy variables $\phi = \phi(x, y)$ and $\tau = \tau(x, y)$ instead of only $\phi = \phi(x, y)$ as in the two space dimensional case. Thus, following the same procedure the three-dimensional form of heat conduction equation (1), i.e.

$$c\rho\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(K\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial U}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial U}{\partial z}\right)$$

can be written finally as

$$\frac{c\rho\partial z}{k\partial t} = \left\{\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)\left(\frac{\partial z}{\partial U}\right)^2 + \frac{\partial^2 z}{\partial U^2}\left(1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right) - 2\frac{\partial^2 z}{\partial x\partial U}\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial U}\right) - 2\frac{\partial^2 z}{\partial y\partial U}\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial U}\right)\right\}\left(\frac{\partial z}{\partial U}\right)^{-2}. \tag{16}$$

An iterative finite difference solution of the steady-state form of this equation in the model of the three-dimensional free boundary problem is also solved in ref. [16].

3. TWO-DIMENSIONAL PROBLEM AND THE TRANSFORMATION

As an example of the application of the above-mentioned transformation to a two-dimensional moving boundary problem, we consider the following. An infinitely long square prism $-1 \leqslant x \leqslant 1$ and $-1 \leqslant y \leqslant 1$, is initially filled with fluid at a fusion temperature, say of unity. The temperature on its surface is kept constant at zero which is below the fusion temperature to ensure that the solidification surface moves inwards.

Because of symmetries about the axes and the diagonal $y = x$, it is sufficient to work in the triangular region R, defined by $R = \{x, y | y \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$. Moreover, it would also be more appropriate to work in the cylindrical polar coordinate system, to make use of a suitable choice of coordinate system for the problem considered above with existing symmetries. Hence, in cylindrical polar coordinates it can be written as $R = \{r, \theta | 0 \leqslant r \leqslant C(\theta),\ 0 \leqslant \theta \leqslant \pi/4\}$, where $C(\theta)$ is the boundary of the prism. Therefore, we mathematically require a solution of the equation

$$U_t = U_{rr} + U_r/r + U_{\theta\theta}/r^2 \quad \text{in } \Omega \tag{17}$$

where Ω is the domain bounded by $r = 1/\cos\theta$, on which $U = 0$ and by the moving interface defined by $r = S(\theta, t)$ on which $U = 1$, for $0 \leqslant \theta \leqslant \pi/4$, $t > 0$.

Also, due to various symmetries, the additional conditions to be satisfied are

$$U_\theta = 0 \quad \text{at} \quad \theta = 0 \quad \text{and} \quad \pi/4. \tag{18}$$

In addition to the interface condition given above, the moving boundary must satisfy the second condition which is known as the latent heat condition. Since, in the example, the liquid phase is always at the uniform temperature $U = 1$ and there is no temperature gradient in the liquid phase, the latent heat condition can be written in more appropriate form, namely

$$\frac{\partial U}{\partial n} = \beta V_n \tag{19}$$

where n is the outward normal to the interface; V_n the velocity of the interface in the direction of n and β a constant depending on the thermal properties of the material undergoing the phase change. But, now, the interface latent heat condition (19) is not in the form suitable for development of analytical or numerical solution of the phase change problems of this kind. Therefore, following Ozisik [17] and in the light of the transformation given in Section 2, we can write equation (19) in a more convenient form as

$$\frac{\partial S}{\partial t} = \frac{1}{\beta}\left(1+\frac{1}{S^2}\left(\frac{\partial S}{\partial \theta}\right)^2\right)\left(\frac{\partial S}{\partial U}\right)^{-1}. \tag{20}$$

Further, we now interchange U and r so that $r = r(U, \theta, t)$ becomes the new dependent variable instead of U. With suitable changes of nomenclature, and using the techniques described earlier equation (17) results in the following differential equation:

$$\frac{\partial r}{\partial t} = \left\{\left(\frac{\partial r}{\partial U}\right)^2 \frac{\partial^2 r}{\partial \theta^2} - \frac{1}{r^2} + \frac{\partial^2 r}{\partial U^2}\left(1+\frac{1}{r^2}\left(\frac{\partial^2 r}{\partial \theta^2}\right)^2\right) - \frac{2}{r^2}\frac{\partial r}{\partial U}\frac{\partial r}{\partial \theta}\frac{\partial^2 r}{\partial \theta \partial U}\left(\frac{\partial r}{\partial U}\right)^{-2} - \frac{1}{r}\right\}. \tag{21}$$

As easily seen, equation (20) now replaces equation (21) which is valid at all other points of the region. Thus, we solve the test problem numerically using equation (21) with the moving boundary condition (20) as well as the consistent transformed boundary conditions in the transformed domain in (U, θ, t)-space.

4. DISCRETIZATION OF TRANSFORMED EQUATIONS

As usual, we work on the U–θ grid by choosing δU and $\delta\theta$ such that $U_i = U_0 + i\delta U$, $i = 0, 1, 2, \ldots, N$ ($U_0 = 0$ and $U_N = 1$) and $\theta_j = \theta_0 + j\delta\theta$, $j = 0, 1, 2, \ldots, M$ ($\theta_0 = 0$ and $\theta_M = \pi/4$). We evaluate the values of r on this grid at successive time steps $t_n = t_0 + n\cdot\delta t$ where t_0 is the time when numerical computations are commenced and δt is the time interval. We also point out that $U_0 + k\cdot\delta U$ is the kth isotherm on which the temperature is U_k in the (r, θ)-plane. Therefore, we calculate the values of r in equation (21) by the usual finite difference and obtain an explicit expression for $r_{i,j}^{n+1}$, the value r at $U = U_0 + i\cdot\delta U$, $\theta = \theta_0 + j\delta\theta$ and t_{n+1} in terms of values already available at $(U_0 + i\delta U,\ \theta_0 + j\delta\theta,\ t_n)$. Now let $\Delta r_{i,j}^n$ denote the displacement of the isotherm U_i along θ_j in the time interval t, i.e.

$$r_{i,j}^{n+1} = r_{i,j}^n + \Delta r_{i,j}^n$$

where from equation (21), $r_{i,j}^n$ is

$$\begin{aligned}\Delta r_{i,j}^n = \Bigg\{&\left(\frac{r_{ij}^n - r_{i-1j}^n}{\delta U}\right)\left(\frac{r_{ij+1}^n - 2r_{ij}^n + r_{ij-1}^n}{(\delta\theta)^2}\right)\frac{\delta t}{(r_{ij}^n)^2} \\ &+ \frac{r_{i+1j}^n - 2r_{ij}^n + r_{i-1j}^n}{(\delta U)^2}\left(1+\frac{\delta t}{r_{ij}^n}\left(\frac{r_{ij}^n - r_{ij-1}^n}{\delta\theta}\right)^2\right) \\ &- \frac{2t}{r_{ij}^n}\left(\frac{r_{ij}^n - r_{i-1j}^n}{\delta U}\right)\left(\frac{r_{ij}^n - r_{ij-1}^n}{\delta\theta}\right) \\ &\times\left(\frac{r_{i+1j+1}^n - r_{i+1j-1}^n - r_{i-1j+1}^n + r_{i-1j-1}^n}{\delta U\cdot\delta\theta}\right) \\ &\times\left(\frac{r_{ij}^n - r_{i-1j}^n}{\delta U}\right)^{-2} - \frac{\delta t}{r_{ij}^n}\Bigg\}\end{aligned}$$

$$i = 1, 2, \ldots, N-1, j = 0, 1, \ldots, M. \tag{22}$$

In a similar manner, the finite-difference form of condition (20) is

$$\Delta r_{Nj} = \frac{\delta t}{\beta}\left(1+\frac{1}{(r_{Nj})^2}\left(\frac{r_{Nj+1}^n - r_{Nj-1}^n}{2\cdot\delta\theta}\right)^2\right)\left(\frac{r_{Nj}^n - r_{N-1j}^n}{\delta U}\right)^{-1}. \tag{23}$$

In addition as the isotherm $U = U_0$ always coincides with the fixed surface of the prism, the condition on the surface is $\Delta r_{0,j}^n = 0$. Thus, if initial data are given of the coordinates of a number of points along each set of isotherms, we can use the approach described here to advance the isotherms at successive time intervals δt.

5. NUMERICAL RESULTS

As our method is not self starting, one has to commence the computation after adopting the initial position of the isotherms from some other sources. However, to generate an initial set of isotherms an analytic or some alternative solution is needed to provide the temperature distribution at some small time t_0. Sometimes, a few steps can be calculated by using a finite-difference form of the original equation on the r, θ grid, then r values obtained by suitable interpolation can be transferred to the U, θ grid and the original method of solution proceeds. In the present case, in order to assess the accuracy of the results obtained from the present method, we also make a start, like Crank and Gupta [7] by taking the initial values of the temperature and interface positions from the one parameter integral method of Poots [18] at $t = 0.0461$.

We have selected $\delta U = 0.1$ and $\delta\theta = \pi/40$ and taken time steps to be $t = 0.0001$ to keep the scheme stable. Table 1 along with the comparative figures from Crank and Gupta [7], Crank and Crowley [8] and Gupta and Kumar [15]. In all four methods, the values corresponding to the distance of the interface on the axis agree well from beginning to end. On the other hand, as far as the figures corresponding to the distance along the diagonal are concerned, the results obtained from the present method agree better than those of Gupta and Kumar [15]. Also, the results tend to agree with those of Crank and Crowley [8] as time goes by.

The complete solidification of the prism is obtained as 0.6187 where the corresponding value from Gupta and Kumar [15] is 0.6302, but values from Crank and Gupta and Crank and Crowley are not available for comparison, since the methods cannot be perused until complete solidification.

It is worth pointing out that to reduce the number of calculations after some time, the displacement of the circular isotherms can be calculated at the points on the axis only. Because, as the isotherm is expected to assume a circular shape gradually near the end of the process, we keep computing the differences between the distances of the interface along the axis and along the diagonal for each isotherm. As soon as it becomes negligible within the desired accuracy, we can fix the centre of the isotherm at the origin. The displacement of the circular isotherm need not be calculated for all values of θ except the axis.

6. CONCLUDING REMARKS

A numerical method for a general multi-dimensional moving boundary problem has been presented. The method proposed in this paper has been illustrated by examining the solidification of a square prism of fluid, however, it is reasonable to expect that the principles can be applied to a wide variety of systems of different geometries and characteristics.

On the other hand, the method can offer appreciable advantages for problems involving variable heat parameters, particularly when they are temperature dependent. The parameters need not be evaluated for the different set of temperatures calculated at each time step at the point of a r–θ grid as in the traditional finite-difference solution of equation

Table 1. Comparison of the x-coordinate of the solid–liquid interface on the x-axis and the diagonal

Time, t	Crank and Gupta [7]	Crank and Crowley [8]	Gupta and Kumar [15]	Present $\delta U = 0.1$, $\delta\theta = \pi/40$ $\delta t = 0.0001$
		On the x-axis		
0.05	0.8125	0.775	0.8125	0.8125
0.10	0.6979	0.676	0.6982	0.6982
0.15	0.6157	0.601	0.6156	0.6156
0.20	0.5473	0.536	0.5463	0.5465
0.25	0.4865	0.477	0.4837	0.4840
0.30	0.4302	0.420	0.4244	0.4246
0.35	0.3766	0.364	0.3663	0.3664
0.40	0.3337	0.308	0.3078	0.3080
0.45	0.2816	0.249	0.2495	0.2496
0.50	—	0.188	0.1894	0.1893
0.55	—	0.119	0.1271	0.1252
0.60	—	—	0.0562	0.0556
		On the diagonal		
0.05	0.6483	0.732	0.6476	0.6478
0.10	0.5812	0.619	0.5642	0.5701
0.15	0.5103	0.536	0.4935	0.5062
0.20	0.4428	0.463	0.4264	0.4385
0.25	0.3948	0.399	0.3642	0.3751
0.30	0.3351	0.342	0.3130	0.3280
0.35	0.2831	0.282	0.2590	0.2751
0.40	0.2332	0.237	0.2176	0.2224
0.45	0.1947	0.188	0.1764	0.1804
0.50	—	0.139	0.1339	0.1371
0.55	—	0.087	0.0899	0.0899
0.60	—	—	0.0398	0.0351

(17). Instead the parameters are evaluated once and for all before the method starts and only for the constant U lines of the U–θ grid.

Moreover, the decision as to whether the transformation should be applied to r or θ (x or y in the Cartesian case) may be influenced by the nature of the boundary conditions and it is naturally problem dependent.

However, as a small drawback, the temperature U should be a single-valued function of the independent variables initially and at any subsequent time. The temperature at any point can only have one value at a given time. Where symmetry does not exist or where use of it might introduce some fresh difficulties this may be overcome by suitable choice of the coordinate system or the choice of an independent variable or other convenient means. But, such methods, for the time being, lie outside the scope of this paper.

Finally, the extension of the numerical method is apparently feasible for resolution of three-dimensional problems and problems with more complicated situations. Nevertheless, this will be left to a later date.

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Laminar natural convection heat transfer from inclined surfaces

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INTRODUCTION

NATURAL convection heat transfer from flat plates is a problem which is of interest in a variety of industrial applications. As a result, these flows have been the subject of numerous theoretical and experimental studies. In addition to the limiting cases of flow adjacent to vertical and horizontal surfaces, the intermediate case of inclined plate heat transfer has also been examined by a number of investigators.

Theoretical predictions of laminar heat transfer correlations can be obtained from similarity solutions for vertical plates as described by Sparrow and Gregg [1] and Yang [2], who followed the classic work by Ostrach [3]. Extension of similarity solutions to inclined surfaces has been discussed by Chen and Yuh [4]. Similarity solutions for a constant flux surface predict a correlation of the form

$$Nu_x = C(Gr_x^*)^{1/5} \tag{1}$$

where the local Nusselt number and the modified local Grashof number are defined by

$$Nu_x = \frac{h_x x}{k} \quad \text{and} \quad Gr_x^* = Gr_x\, Nu_x = \frac{g\beta q_w x^4}{k\nu^2}. \tag{2}$$

A theoretical prediction for heat transfer to room temperature water can be obtained from the similarity solution for a Prandtl number of 6.14 (corresponding to water at 25°C). In terms of Ra_x^*, the modified Rayleigh number, the predicted relation can be shown [5] to be

$$Nu_x = 0.587(Ra_x^*)^{1/5}. \tag{3}$$

Experimental studies of heat transfer adjacent to inclined surfaces began with work conducted with isothermal surfaces. Rich [6] was the first to show that heat transfer coefficients could be correlated for inclined surfaces by using only the component of gravity parallel to the surface in the Grashof number ($g \cos\theta$ instead of g where θ is the angle from vertical). Rich's result, which was obtained for an isothermal surface inclined at up to 40° from vertical has been supported by numerous subsequent investigators working with both isothermal surfaces (e.g. refs. [7, 8]) and with uniform heat flux surfaces (e.g. refs. [9, 10]).

Available correlations for uniform heat flux inclined surfaces stem primarily from the studies by Vliet [9]. Vliet arrived at a correlation of

$$Nu_x = 0.60(Gr_x^* \cos\theta\, Pr)^{0.20} \tag{4}$$

using a heated foil and both water and air as working fluids. Vliet made measurements of temperature differences as small as 3°F (1.7°C) and estimated his measurement error for the Nusselt number at 5% for high heat fluxes and 15% for low heat fluxes. The lowest Rayleigh numbers measured were in the neighborhood of $Ra_x^* = 6\times10^6$. The highest heat flux measurements extended to $Ra_x^* = 2\times10^{15}$; well into the turbulent flow region. Inclinations measured ranged from vertical to 60° from vertical. Another set of data for inclined surfaces was obtained using a heated foil in air [10], and resulted in a correlation of

$$Nu_x = 0.55(Gr_x^* \cos\theta\, Pr)^{0.20}. \tag{5}$$

In this instance, measurements for upward facing plates were made only for 30° from vertical and for vertical. The laminar data extended down to modified Rayleigh numbers as low as 5×10^8. Since the experiments were conducted in air, corrections were required for radiative and conductive losses of heat. These corrections amounted to 18–23% of the heat dissipated. Deviation of the Nusselt numbers from the recommended correlation appears to be in the range of 10–20% for the upward facing inclined data.

Here, we describe measurements of laminar inclined surface heat transfer data which were obtained over a wider range of experimental conditions and with more precision than has been previously reported. The experimental scatter of Nusselt numbers measured by the techniques presented in this investigation was typically less than 1–2%. The results described are an offshoot of a set of studies which were directed toward an improved understanding of atmospheric mountain slope flows [5].

With measurements of greater precision, the accuracy and range of applicability of the similarity solutions can be more adequately assessed. In particular, the validity of applying the vertical plate correlations to near-horizontal inclinations by a $\cos\theta\, Ra_x^*$ correction is addressed. In addition, the low Rayleigh number behavior of the heat transfer coefficient is described. Isothermal data reported in ref. [11] indicate that there is a lower limit to the linear region of heat transfer correlations associated with a change of regime from convection heat transfer to conduction. Such measurements for uniform flux surfaces require a high degree of precision in determining the surface to bulk temperature difference. As Holman [11] pointed out, experimental scatter of $\pm 20\%$ is typical for these types of experiments which has made it difficult to address these problems in previous investigations.

EXPERIMENTAL DESIGN

In this section we begin by briefly describing the equipment used for these experiments. Drawings and more extensive details of the construction are available elsewhere [5] and are not included here. The heated surface used for these experiments was 0.3048 m long and 0.1524 m wide and consisted of a 0.0254 mm stainless steel foil backed by 0.0191 m thick extruded polystyrene insulation and a 0.0127 m acrylic plate. The heated foil (made up from stainless steel #304

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